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Ergodic interpretation of integral hydrodynamic invariants

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We generalize to higher dimensions the result of V. Arnold that the integral Hopf invariant (or total helicity) of a closed two-form on a three-dimensional manifold coincides with the asymptotic linking number of its vorticity field. For the higher-dimensional case the integral invariant of Hopf–Novikov type for a set of forms turns out to be equal to an asymptotic "multilinking number" of the corresponding kernel foliations. In particular, the notions and main properties of "generic" and "nongeneric" multilinking numbers are described.

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The classical Hopf invariant for $S^3 \to S^2$ mappings has two well-known definitions: a geometrical one (as the linking number of two curves which are inverse images of two arbitrary points of S^2) and an integral one (as the value $\int \alpha \wedge d^{-1}\alpha$ for any two-form α on S^3 which is the pull back of a generator of the group $H^2(S^2, \mathbb{Z})$). In ref. [1], Arnold generalized the geometric invariant to the case of any divergence-free vector field on S^3. It turned out that the integral Hopf invariant of any closed two-form on S^3 is equal to the asymptotic linking number of the trajectories (or "helicity") of the kernel fields for this two-form. This paper is devoted to the ergodic interpretation of the analogous integral invariants of two-forms on manifolds of higher dimensions.

For odd-dimensional manifolds, we consider the geometrical meaning of invariants of the type $\int d^{-1}\alpha \wedge \beta \wedge \cdots \wedge \omega$ to be the multilinking number of foliations corresponding to these forms $\alpha, ..., \omega$. These integrals appear in the group theoretic approach to the theory of an ideal and barotropic fluid [3]. Unlike the three-dimensional case, for which one can define the asymptotic linking number for any pair of trajectories, in the general case it is necessary to consider the linkage of one trajectory with the whole foliation of codimension 2 (see sections 2, 3). For even-dimensional manifolds the interpretation of Novikov invariants [4] is described to be an averaged nongeneric linkage (see section 4).

In fact, here we present a sort of dictionary that translates facts on differential forms into facts concerning averaged linking properties of their kernels. For singular forms concentrated on compact surfaces, these two languages coincide. It would be interesting to find an analogous dictionary for various characteristic classes of foliations (cf. refs. [6,2]).

Notice, that the integrals described define multivalued functions on the set of mappings of a sphere into a given manifold [4]. Homotopically invariant functionals generate an "integer lattice" in the space of integrals and so their role is equivalent to the role of the classical Hopf invariant among the asymptotic ones.

1. Hopf–Novikov integrals

In this section we recall the known facts concerning integrals of smooth two-forms on manifolds.

1.1. INVARIANTS OF TWO-FORMS ON ODD-DIMENSIONAL MANIFOLDS

Let M^{2m+1} be a compact connected manifold without a boundary, and, moreover, $H_1(M)=H_2(M)=0$. Denote closed two-forms on M by $\alpha, \beta, \ldots \in \Omega^2(M)$, $d\alpha = d\beta = \cdots = 0$. Let $d^{-1}\alpha$ be an arbitrary primitive one-form for α (two different primitives differ by a differential of a function on M). There exists the obvious

Proposition 1.1. *For the set $\alpha, \beta, \ldots, \omega$ of $m+1$ closed two-forms on M^{2n+1}, the quantity $I(\alpha, \ldots, \omega) = \int_M d^{-1}\alpha \wedge \beta \wedge \cdots \wedge \omega$ depends on $\alpha, \ldots, \omega$ only and does not depend on the particular choice of $d^{-1}\alpha$.*

Remark 1.2. The value of $I(\alpha, \ldots, \omega)$ does not change under a permutation of the arguments.

Remark 1.3. Invariants of the type $I(\alpha, \alpha, \ldots, \alpha)$ are the first integrals of the motion of a $(2m+1)$-dimensional ideal incompressible fluid [3].

Let the manifold M^{2m+1} be equipped with a volume form μ [$\mu \in \Omega^{2m+1}(M)$]. Then the statements on closed two-forms can be reformulated in terms of divergence-free $(2m-1)$ vector fields on M (denoted by script letters), which are defined by the conditions $i_{\mathcal{A}}\mu = \alpha$, $i_{\mathcal{B}}\mu = \beta$, etc. The "divergence-free" property of $\mathcal{A}$ means that $d\alpha = 0$.

Theorem 1.4 [1]. *For M^3 the quantity $I(\alpha, \beta)$ coincides with the average asymptotic linking number of vector fields $\mathcal{A}$ and $\mathcal{B}$ which are the vorticities of two-forms α and β (i.e., $i_{\mathcal{A}}\mu = \alpha$ and $i_{\mathcal{B}}\mu = \beta$).*

(The definition of the average linking number is given below and in ref. [1].)

The ergodic interpretation of the invariant $I(\alpha, \beta, ..., \omega)$ for $n>1$ is proposed in sections 2, 3. Notice that in general the multivector field $\mathscr{A}$ defines a foliation (and a measure on it) only if the rank of the two-form α is at most 2 on M. Indeed, in this case the field $\mathscr{A}$ is the kernel field for α, and the kernel field for a closed two-form is completely integrable (Frobenius integrability criterion). If $rk_x\alpha>2$ at a certain point $x\in M$, then $\mathscr{A}(x)$ is an indecomposable $(2m-1)$ vector and does not define any subspace in T_xM.

1.2. NOVIKOV INVARIANTS OF TWO-FORMS ON EVEN-DIMENSIONAL MANIFOLDS

Let M^4 be an arbitrary four-dimensional manifold, and let α, β be exact two-forms satisfying the following conditions on their external squares:

$$\alpha\wedge\alpha=\beta\wedge\beta=0\,, \tag{1}$$

and product:

$$\alpha\wedge\beta=0\,. \tag{2}$$

Proposition 1.5 [4]. *The integrals* $\mathscr{I}(\alpha,\alpha,\beta)=\int_M d^{-1}\alpha\wedge\alpha\wedge d^{-1}\beta$ *and* $\mathscr{I}(\alpha,\beta,\beta)=\int_M d^{-1}\alpha\wedge\beta\wedge d^{-1}\beta$ *do not depend on the choice of* $d^{-1}\alpha$ *and* $d^{-1}\beta$.

Remark 1.6. In ref. [4] Novikov represented the set of invariants for arbitrary-dimensional M; we avoid the more complicated formulas and consider the case M^4 just for illustration.

Let the manifold M^4 be equipped with a volume form μ. Condition (1) means that two-forms α and β have rank ≤ 2 over M, i.e., these forms define two-dimensional foliations $\mathscr{A}$ and $\mathscr{B}$ (with measures on them). Condition (2) provides a nongeneric position of them, namely, their intersection is a one-dimensional foliation (and, moreover, their sum defines a three-dimensional one [1]).

1.3. CRUCIAL POINT OF THE INTERPRETATION

In fact, the exposition given here represents a reformulation and interpretation of the concepts involved in the following language: instead of "closed two-forms", we say their "kernel fields" (for forms of rank ≤ 2, the kernel fields form foliations of codimension 2), so the operations d^{-1} and $\wedge$ correspond to transferring from the surfaces to the films bounded by them and to their intersections, respectively. Finally, the integration $\int_M$ is the summation of the intersection points with the corresponding signs.

The intersection of a manifold with a film bounded by another manifold gives us their linking number, so after all the invariants under consideration are certain average linking numbers.

2. Geometric meaning of invariants on odd-dimensional manifolds

In this section we describe the ergodic interpretation of the invariant $I(\alpha, ..., \omega)$, where the rank of one of the two-forms $\alpha, ..., \omega$ is at most 2.

2.1. THE LINKING OF A FOLIATION OF CODIMENSION 2 WITH A CURVE

A closed two-form α of rank ≤ 2 determines a (nonregular) kernel foliation of codimension 2 on the manifold M^n. If M is equipped with a volume form μ, then this foliation is the field of $(n-2)$ vectors $\mathscr{A}$ such that $i_{\mathscr{A}}\mu = \alpha$ (i.e., of the kernels of α). Being a kernel field of a closed two-form, this field is integrable.

Definition 2.1. The average linking number of a closed curve $\Gamma \subset M$ and the foliation $\mathscr{A}$ is the flow of the two-form α through an arbitrary film $\partial^{-1}\Gamma$ bounded by Γ:

$$k(\Gamma, \mathscr{A}) = \int_{\partial^{-1}\Gamma} \alpha = \int_{\Gamma} d^{-1}\alpha \,. \tag{3}$$

Remark 2.2. For any manifold M^n [for which $H^1(M) = H^2(M) = 0$] there exists a "linking form" $L \in \Omega^1(M) \times \Omega^{n-2}(M)$ such that for given arbitrary nonintersecting compact one- and $(n-2)$-dimensional submanifolds Γ and Θ in M, their "usual" linking number is equal to $\int\int_{\Gamma \times \Theta \subset M \times M} L$ (see refs. [3,5]).

The following proposition motivates the definition of $k(\Gamma, \mathscr{A})$.

Proposition 2.3. *The number $k(\Gamma, \mathscr{A})$ coincides with the linking number of Γ and the separated fibers of $\mathscr{A}$ evaluated with the help of the "linking form" L and averaged over M.*

Proof.

$$\iint_{\Gamma \times \mathscr{A}} L = \iint_{\Gamma \times M} i_{\mathscr{A}} L \wedge \mu = \iint_{\Gamma \times M} L \wedge i_{\mathscr{A}}\mu = \iint_{\Gamma \times M} L \wedge \alpha = \int_{\Gamma} d^{-1}\alpha \,.$$

[The first identity is the definition of $\int\int_{\Gamma \times \mathscr{A}} L$, the last one is the main property of L: the "linking form" acts on the exact differential forms just like the operator d^{-1} (see ref. [3]).] □

Remark 2.4. For a foliation $\mathscr{A}$ with compact nonsingular fibers, the number $k(\Gamma, \mathscr{A})$ is exactly the average linking number of the curve Γ and every fiber.

Indeed, the contribution of each fiber is propositional to its measure given by $\mathscr{A}$ and its linking number with Γ. For instance, let there exist a projection $\pi : M^n \to N^2$ and let α be a pull back of some area form ν on N. Then

$$k(\Gamma, \mathscr{A}) = \int_{\Gamma} d^{-1}\alpha = \int_{\pi(\Gamma)} d^{-1}\nu = \int_{\partial^{-1}\pi(\Gamma)} \nu ,$$

i.e., k is the oriented area inside the projection $\pi(\Gamma)$ of the curve Γ.

Remark 2.5. For $n=3$ one can define not only the average linking number of Γ with $\mathscr{A}$, but also the asymptotic linking number of Γ with separate fibers of $\mathscr{A}$ [1].

In this case $\mathscr{A}$ is the vorticity vector field curl α of the two-form α: $i_{\mathrm{curl}\,\alpha}\mu = \alpha$. A segment of any trajectory (of this vector field) considered for some large time T and closed by a short path has a certain linking number with the curve Γ. The time average of this number is the asymptotic linking number of this trajectory and Γ.

If the fibers of $\mathscr{A}$ are noncompact and $n > 3$, generally speaking, it is impossible to define the asymptotic linking number of separate fibers with Γ. For the multi-dimensional case there is no satisfactory definition of the system of short paths and of canonical "time" on the fibers.

2.2. ASYMPTOTIC LINKING NUMBER OF A VECTOR FIELD AND A FOLIATION

Let $g_v^t x$ be the trajectory of the vector field v issuing from the point $x \in M$. We select a large number T and close the segment $g_v^t x$ $(0 \leqslant t \leqslant T)$ of this trajectory by a certain "short" path, so that we obtain a closed curve $\Omega_v^T(x)$.

Definition 2.6. An asymptotic linking number $k(x, v, \mathscr{A})$ of the trajectory $g_v^t x$ of the vector field v issuing from the point $x \in M$ is defined as the limit

$$k(x, v, \mathscr{A}) = \lim_{T \to \infty} \frac{k(\Omega_v^T(x), \mathscr{A})}{T} . \tag{4}$$

It turns out that this limit exists almost everywhere and is independent of the system of "short" paths (the axioms for this system are given in ref. [1]).

Definition 2.7. The average linking number $\bar{K}(v, \mathscr{A})$ of a vector field v and a foliation $\mathscr{A}$ on M (equipped with a volume form μ) is $\bar{K}(v, \mathscr{A}) = \int_M k(x, v, \mathscr{A})\mu$.

Now we are able to give an ergodic interpretation of the Hopf-type integral invariant $I(\alpha, ..., \omega)$ for a set of $m+1$ closed two-forms α, ..., ω on an odd-dimensional manifold M^{2m+1} with volume form μ.

Theorem 2.8. *Let the rank of one of the forms (for example, α) be at most 2. Then $I(\alpha, \beta, ..., \omega) = \int_M d^{-1}\alpha \wedge \beta \wedge \cdots \wedge \omega$ coincides with the average linking number of the vector field* $\mathrm{curl}(\beta, ..., \omega)$ *and the foliation* $\mathscr{A}$: $I(\alpha, ..., \omega) = \bar{K}(\mathrm{curl}(\beta, ..., \omega), \mathscr{A})$, *where the fields* $\mathrm{curl}(\beta, ..., \omega)$ *and* $\mathscr{A}$ *are defined by* $i_{\mathrm{curl}(\beta, ..., \omega)}\mu = \beta \wedge \cdots \wedge \omega$ *and* $i_{\mathscr{A}}\mu = \alpha$, *respectively.*

Remark 2.9. Certainly, for the definition of the flow of α in definition 2.1, and consequently for this theorem, rank α is not important: in the general case, we would consider the linkage with an $(n-2)$-vector field instead of an $(n-2)$-dimensional foliation.

Proof of theorem 2.8. The vector field $v = \mathrm{curl}(\beta, ..., \omega)$ is divergence-free, because the form $i_v\mu = \beta \wedge \cdots \wedge \omega$ is closed. Hence, the field v preserves the volume form μ, and we may apply the Birkhoff ergodic theorem to the flow of v.

Replacing the time average in the definition of $\bar{K}$ by the space average, we obtain the following:

$$\bar{K}(v, \mathscr{A}) = \int_M \left(\lim_{T\to\infty} \frac{1}{T} k(g_v^T x, \mathscr{A})\right)\mu = \int_M \left(\lim_{T\to\infty} \frac{1}{T}\int_0^T i_v d^{-1}\alpha\right)\mu$$
$$= \int_M (i_v d^{-1}\alpha)\mu = \int_M d^{-1}\alpha \wedge i_v\mu$$
$$= \int_M d^{-1}\alpha \wedge \beta \wedge \cdots \wedge \omega = I(\alpha, ..., \omega) .$$

(The second identity follows immediately from definition 2.1, because the integral of $d^{-1}\alpha$ measures the linkage number with $\mathscr{A}$.) □

3. Multilinking number of a few foliations

Closed two-forms $\alpha, ..., \omega$ with rank $\leqslant 2$ determine the foliations and the invariant $I(\alpha, ..., \omega)$ measures the complicity of the mutual multilinkage of their fibers.

3.1. DEFINITION OF THE MULTILINKING NUMBER

The usual linking number is defined for any pair of surfaces P^s and Q^l in $\mathbb{R}^n$ with $s+l=n-1$, $P \cap Q = \varnothing$ (i.e., this is a bilinear form on the space of nonintersecting submanifolds of appropriate dimensions). We define the multilinking number as a multilinear form on the space of r-tuples of submanifolds $(P_1, ..., P_r)$ such that $\sum_{i=1}^{r} \operatorname{codim}(P_i) = n+1$ and $\bigcap_{i=1}^{r} P_i = \varnothing$.

For instance, one can link three circles on a plane (fig. 1a) or two spheres and one circle in three-space (fig. 1b).

Definition 3.1. The multilinking number of r oriented closed submanifolds $P_1, ..., P_r$ in $\mathbb{R}^n(S^n)$ satisfying the above conditions is the number of intersection points (with the corresponding signs) of a film bounded by one of these surfaces P_i with all the other submanifolds.

If these submanifolds are equipped by a transverse orientation then all films bounded by them and their intersections are also oriented. So the signs of the intersection points are well defined from comparison of their transverse orientations with the orientation of $\mathbb{R}^n(S^n)$.

Proposition 3.2. *The absolute value of the multilinking number of the set $P_1, ..., P_r$ does not depend on the choice of the initial surface P_i.*

Proof. Suppose we calculate the multilinking numbers starting with P_i or P_j. Consider arbitrary films $\partial^{-}P_i$ and $\partial^{-1}P_j$ and look at the complete intersection of the remaining $r-2$ surfaces with both films at once. This intersection is a one-dimensional submanifold in M^n. This submanifold connects two sets of points, determining the multilinking numbers in both cases. Therefore these sets of points are homologically the same, and the corresponding multilinking numbers coincide.

Another way to prove the proposition is to notice that the multilinking number coincides with the usual linking number of the submanifold $P_1 \times P_2 \times \cdots \times P_r \subset M \times \cdots \times M$ with the diagonal $\Delta \subset M \times \cdots \times M$ $[\Delta = (x, ..., x), x \in M]$.

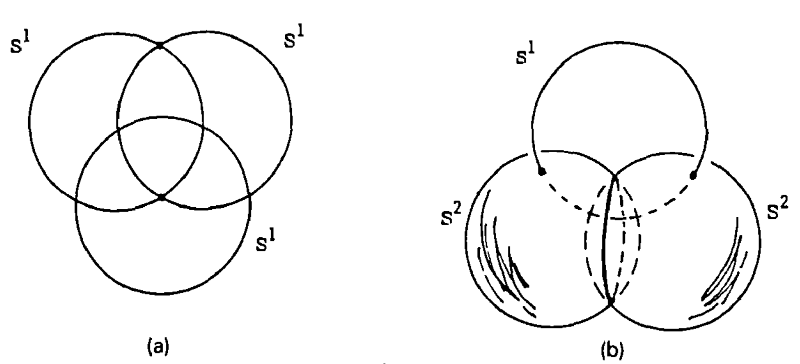

Fig. 1.

Remark 3.3. This multilinking number can be defined as the usual linking number of one of the surfaces with the intersection of all others. Example: the linkage of $m+1$ surfaces of codimension 2 in $\mathbb{R}^{2m+1}$ can be reduced to the linkage of one of them with the intersection curve of the rest.

3.2. ASYMPTOTIC MULTILINKING NUMBER

Suppose that $m+1$ closed two-forms $\alpha, ..., \omega$ define the foliations of codimension 2 in M^{2m+1}.

Definition 3.4. The asymptotic linking number of one of these foliations (of codimension 2) with (one-dimensional) foliation being the intersection of the remaining m is called the asymptotic multilinking number of the given foliations.

Notice that for the forms, $\beta, ..., \omega$ the intersection is determined by the vector field curl$(\beta, ..., \omega)$. The following theorem is just a reformulation of theorem 2.8.

Theorem 3.5. *The integral $I(\alpha, ..., \omega)$ is equal to the asymptotic multilinking number of the foliations given by the forms $\alpha, ..., \omega$.*

4. Geometric meaning of Novikov's invariants

4.1. NONGENERIC LINKAGE

In the preceding section we saw that submanifolds may possess a nontrivial multilinking number only if the sum of their codimensions is $m+1$. To describe the ergodic meaning of Novikov's invariants, let us extend the concept of multilinking: we discard the codimension condition if it is compensated by new nongeneric intersection conditions. For instance, two circles S^1 and a sphere S^2 cannot be linked in $\mathbb{R}^3$ (see fig. 2, one can untie any configuration of them not passing through triple points). However, if these two circles are meridians of the same ball (and so their intersection consists of two points, i.e., it is S^0), the linkage

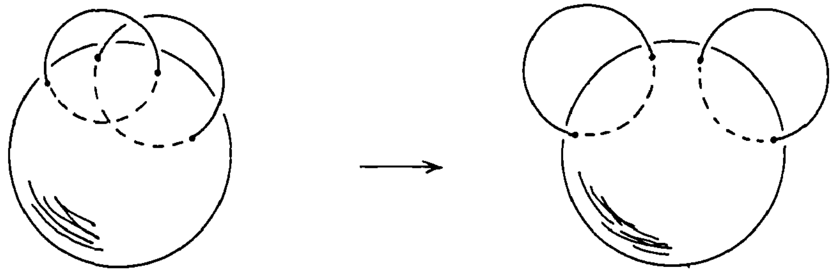

Fig. 2.

may be nontrivial (fig. 3).

In the definition of invariants $\mathscr{I}$, conditions of the type $\alpha \wedge \beta = 0$ provide the nongeneric intersections of the fibers corresponding to α and β.

4.2. ERGODIC INTERPRETATION

Recall the definition of the invariants for a four-dimensional manifold:

$$\mathscr{I}(\alpha, \alpha, \beta) = \int_M d^{-1}\alpha \wedge \alpha \wedge d^{-1}\beta, \qquad \mathscr{I}(\alpha, \beta, \beta) = d^{-1}\alpha \wedge \beta \wedge d^{-1}\beta,$$

where exact two-forms α and β are such that $\alpha \wedge \alpha = \beta \wedge \beta = \alpha \wedge \beta = 0$.

Definition 4.1. The vector field curl L for the three-form L on M^4 (equipped with a volume form μ) defined by $i_{\mathrm{curl}\, L}\mu = L$ is called the vorticity field of L.

Theorem 4.2. *The invariant* $\mathscr{I}(\alpha, \alpha, \beta)$ $[\mathscr{I}(\alpha, \beta, \beta)]$ *coincides with the average linking number of the foliation* $\mathscr{A}$ *of the form* α *[the foliation* $\mathscr{B}$ *of the form* β*] with the vorticity field* $\mathrm{curl}(d(d^{-1}\alpha \wedge d^{-1}\beta))$.

Proof. The three-form $d(d^{-1}\alpha \wedge d^{-1}\beta)$ is closed, hence its vorticity vector field is divergence-free, and again we may apply the Birkhoff ergodic theorem:

$$\bar{K}(\mathrm{curl}(d(d^{-1}\alpha \wedge d^{-1}\beta)), \mathscr{A}) = \int_M d^{-1}\alpha \wedge i_{\mathrm{curl}(d(d^{-1}\alpha \wedge d^{-1}\beta))}\mu$$

$$= \int_M d^{-1}\alpha \wedge d(d^{-1}\alpha \wedge d^{-1}\beta) = \mathscr{I}(\alpha, \alpha, \beta)\,. \quad \square$$

Corollary 4.3. *The average multilinking number turns out to be independent of the choice of primitive one-forms* $d^{-1}\alpha$ *and* $d^{-1}\beta$, *though the vorticity field* $\mathrm{curl}(d(d^{-1}\alpha \wedge d^{-1}\beta))$ *(in theorem 4.2) does depend on this choice.*

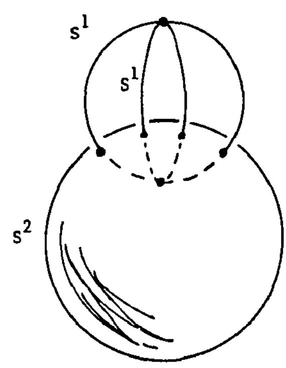

Fig. 3.

Remark 4.4. The vorticity field of the three-form $d(d^{-1}\alpha \wedge d^{-1}\beta)$ is tangent to the three-dimensional foliation which is the sum of the foliations $\mathscr{A}$ and $\mathscr{B}$. Roughly speaking, the quantities $\mathscr{I}(\alpha, \alpha, \beta)$ and $\mathscr{I}(\alpha, \beta, \beta)$ are the asymptotic linking numbers of the one-dimensional foliation of the intersections $\mathscr{A} \cap \mathscr{B}$, with each of the foliation $\mathscr{A}$ and $\mathscr{B}$ determined by α and β.

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